Discipline: **Physics**
Subject: **Electromagnetic Theory**
Unit 35:
Lesson/ Module: **Units and Dimensions**

Author (CW): **Prof. V. K. Gupta**
Department/ University: **Department of Physics and Astrophysics, University of Delhi, New Delhi-110007**

Contents

Learning Objectives:

From this module students may get to know about the following:

1. *Reason for so many systems of units and dimensions in electromagnetic phenomena.*
2. *Various equations of electromagnetism written in a system-independent way.*
3. *Various constants of proportionality that enter into these equations.*
4. *Some of the many systems which were in use and the values of these constants in those systems.*
5. *Table for conversion of equations from one to the other system for the two which are still widely in use.*

35. Units and Dimensions

35.1 Introduction

When dealing with physical quantities we have to employ a system of units and dimensions to measure them. There is of course a plethora of units employed, some in general and some for specific purposes. They can sometimes become confusing; but no where are they more so than in electromagnetic theory, where historically a large number of different units (and dimensions) have been employed. Even now many units are in common use. These are

(i) The international system of units. (SI)
(ii) Gaussian system of units.
(iii) Heaviside-Lorentz system of units.
(iv) Various "natural" system of units.
(v) esu system
(vi) emu system

The SI system is the modern metric system and is the preferred system now. In the context of electromagnetism it is called the MKSA (meter, kilogram, second, ampere) or the rationalized MKSA system.

The other system still commonly used in electromagnetism is the Gaussian system especially in cosmology, astrophysics, particle physics and other fundamental studies.

Heaviside-Lorentz and the "natural" system of units are still often used in quantum electrodynamics, though not much in classical electrodynamics. In the natural system employed in quantum electrodynamics we take $c = \hbar = 1$, and avoid writing a large number of these factors which appear in most quantum electrodynamic calculations. These factors are restored at the end from dimensional considerations.

Let us try to understand the problem of multiplicity of system of units by considering the well-known phenomenon of gravity. In mechanical phenomena there are three fundamental dimensions – mass, length and time, and all other mechanical quantities can be expressed in terms of these. It is an experimentally established fact that the force of gravity between two bodies is directly proportional to their masses and inversely proportional to the square of distance between them:

$$F \propto \frac{m_1 m_2}{r^2} \Rightarrow F = G\frac{m_1 m_2}{r^2}$$

Since the dimensions of mass, length and time are fixed, the units of force follow from them in terms of its definition. (Force = mass × acceleration. There is a problem here also but let us not delve into that.) The dimensions of the constant G are now fixed $[FL^2/M^2]$. The magnitude has, of course, to be determined by actual measurements. Let us have a little digression here. The force is the thing which enters into the law of inertia. There could be various origins of force between two objects. It depends on certain property of those objects called "charge". In this sense the electrostatic force is due to the "electric charge *e*" of the body; the gravitational force is due to the "gravitational charge *g*" of the body and so on. It is again an experimental observation that the gravitational charge of a body equals (actually, proportional to) the inertial mass of the body; the quantity that enters into the law of inertia. In this spirit the law of gravitation should be rewritten as

$$F = G' \frac{g_1 g_2}{r^2}$$

Experimentally when we measure force, we can obtain only the value of $G' g_1 g_2$ and not of G' and g individually. Conceptually, G' is a property of the force and g of the body. Therefore it will be nice to separate the two and define their units and dimensions independently. Further, the concept of field, the force per unit charge $\frac{F}{g_1} = G' \frac{g_2}{r^2}$, is one of the most important and fruitful concepts in physics. To assign dimensions to the field, any field, we must assign dimensions to G' and g independently. That is where many different systems of units (and dimensions) arise. The problem did not occur in mechanics because g happened to be proportional to m; the same units could be (and were) chosen for g as for m. As a result only one new quantity G was introduced and its dimensions got automatically fixed.

Historically, the two phenomena, electrostatics and magnetostatics developed independently and had their own system of units. Till then the problem was still manageable. When the connection between the two was discovered, the confusion became worse confounded because now the two systems had to be reconciled properly.

35.2 System-independent relations

The starting point of each system of units has been the force between objects; in the present context we have the force between two charges or a force between two currents. Historically the latter was written as force between two magnetic monopoles. Let us begin with the Coulomb force between two charged particles. Experimentally it is known to be inversely proportional to the square of distance between two point charged particles. In the light of the discussion above, we write it as

$$F_e = k_e \frac{q_1 q_2}{r^2} \qquad (1)$$

q_1 and q_2 are the "electric charges" of the two particles and k_e is a measure of the strength of the force, a property of the nature of force. The system of units will depend on how we assign dimensions to the two quantities, the charge q and the *coupling constant* k_e.

Another experimental observation is the existence of a force between two current carrying conductors. Two infinitely long conductors, carrying currents I_1 and I_2 respectively experience a force which is proportional to the two currents and inversely proportional to the distance between them. The force per unit length of the conductors is given by the Ampère's law of force

$$F_{mag} = 2k_m \frac{I_1 I_2}{r} \tag{2}$$

This is the basic equation of magnetostatics. The force is due to the interaction of the current in one wire and the magnetic field produced by the other. It thus links electric and magnetic phenomena. The constant factor "2" in the equation is only for convenience in defining the units as we will see.

Unlike the case of gravity, the units of current or charge were not already fixed from some other defining equation. Equation (1) can fix the units of ($k_e q_1 q_2$) but not of k_e and q individually. Similarly equation (2) can fix the units of $k_m I_1 I_2$ but not of k_m and I individually. The units and dimensions of k_e and k_m must be fixed by hand, and that is where the arbitrariness lies.

However, the situation is not totally chaotic. The dimensions of the two constants k_e and k_m are related to each other and only one of them can be chosen arbitrarily. The left hand side of equation (1) has the dimensions of force and that of equation (2) has dimensions of force/length. So their ratio has dimensions of length. On the right hand side, charge and current are related; current is rate of flow of charge and so has the dimensions of charge/time. So considering the dimensions, we have

$$[\frac{F_e}{F_{mag}}] = [L] = [\frac{k_e Q^2}{L^2}\frac{L}{k_m I^2}] = [\frac{k_e}{k_m}\frac{T^2}{L}] \Rightarrow [\frac{k_e}{k_m}] = [L^2 T^{-2}] = [v^2] \tag{3}$$

The ratio k_e/k_m has the dimensions of (velocity)^2. Thus, though in any given system of units dimensions of one can be chosen arbitrarily, those of the other are then fixed.

Both F_e and F_{mag} can be measured experimentally; this fixes the numerical value of the ratio k_e/k_m. Measurements in the nineteenth century gave values for the ratio which were very close to the then known value of the velocity of light in vacuum, c. This closeness could not be fortuitous; it suggested that the phenomenon of light was electromagnetic in nature. In fact now it is known that the ratio $\sqrt{\frac{k_e}{k_m}}$ is exactly equal to the velocity of light in vacuum and is often simply replaced by c:

$$\sqrt{\frac{k_e}{k_m}} = c \tag{4}$$

Since current is defined as the rate of flow of charge, the law of conservation of charge is expressed as the continuity equation

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{5}$$

where ρ is the charge density and $\vec{J}$ the current density.

35.2.1 Enters the magnetic field

We now bring the magnetic field into the picture. The electric field at a point is defined as the force per unit charge:

$$\vec{E} = \frac{\vec{F}_e}{q} = k_e \frac{q}{r^2} \qquad (6)$$

and is thus fixed the moment k_e is fixed. However, an electric current gives rise to a magnetic field (which makes the magnetic needle move) and all we know from equation (2) at this stage is that the magnetic field produced is proportional to the current. Like the electric field, the magnetic field is proportional to force/current and leaves room for another arbitrary constant of proportionality:

$$B \propto \frac{F_{mag}}{I} = 2k_B k_m \frac{I}{r} \qquad (7)$$

where k_B is another proportionality constant to be chosen arbitrarily. Once the dimensions and units of k_B are chosen, those of B are fixed. These equations also give a relation between the units of E and B:

$$[B] = [k_B k_m I / r] = [k_B k_m \frac{q}{rt}] = [k_B k_m \frac{Er}{k_e t}] = [k_B \frac{E}{c}] \qquad (8)$$

Now look at the Lorentz force on a charged particle in an external electromagnetic field. The magnetic part of the force is proportional to $\vec{v} \times \vec{B}$, the constant of proportionality will depend on the system of units chosen. So

$$\vec{F}_l = q[\vec{E} + \alpha \vec{v} \times \vec{B}]$$

From equation (8), $(\vec{v} \times \vec{B})/k_B$ has dimensions of $\vec{E}$, and therefore, demanding that all the terms in an equation must have the same dimensions, we find that $\alpha = 1/k_B$, so that

$$\vec{F}_l = q[\vec{E} + \frac{1}{k_B} \vec{v} \times \vec{B}] \qquad (9)$$

Starting from Coulomb's law, once we fix the dimensions and numerical value of k_e, the dimensions and units of charge are fixed. Then from equation (4) k_m necessarily has the value k_e/c^2. Both, its dimensions and numerical value are fixed. Equation (2) then fixes the force between two currents. Next we must choose a value for k_B. This fixes the dimensions of B. Equation (7) then fixes the magnitude of B from that of current.

So far we have considered stationary phenomena only – electrostatics and magnetostatics. The dynamical relation between the electric and magnetic phenomena is provided by Faraday's law. The law says that a changing magnetic field induces an *emf* in a loop and that the induced *emf* is proportional to the rate of change of magnetic flux. The law is written in a differential form as

$$\vec{\nabla} \times \vec{E} + k_f \frac{\partial \vec{B}}{\partial t} = 0 \qquad (10)$$

This introduces yet another constant in the theory. However, once k_e and k_B are fixed, everything in the above equation is fixed except k_f. Thus the constant k_f must be related to k_e and k_B. Let us elucidate on that by looking at the full set of Maxwell's equations.

35.3 Maxwell's equations

Let us look at the Maxwell equations in their differential form, which are obtained from Coulomb's law, Ampere's law and Faraday's law, and the fact that there are no magnetic monopoles, by the application of theorems of vector analysis. The first one is Gauss law which is a direct consequence of Coulomb's law, and is written as

$$\vec{\nabla} . \vec{E} = 4\pi k_e \rho \qquad (11)$$

The second equation is Ampere's law. Along with the displacement current term introduced by Maxwell, it reads

$$\vec{\nabla} \times \vec{B} = 4\pi k_B k_m \vec{J} + \beta \partial \vec{E} / \partial t \qquad (12)$$

The factor $k_B k_m$ comes from equation (7) and β is determined as follows: Take the divergence of equation (12), use continuity equation (5) for $\vec{\nabla} . \vec{J}$ and Gauss law, equation (11) to obtain

$$0 = -4\pi k_B k_m \frac{\partial \rho}{\partial t} + \beta 4\pi k_e \frac{\partial \rho}{\partial t} \qquad (13)$$

so that

$$\beta = k_B k_m / k_e \qquad (14)$$

On substituting for β from equation (12), the Ampere's law takes the form

$$\vec{\nabla} \times \vec{B} - \frac{k_B k_m}{k_e} \frac{\partial \vec{E}}{\partial t} = 4\pi k_B k_m \vec{J} \qquad (15)$$

Then we have the Faraday's law, equation (10)

$$\vec{\nabla} \times \vec{E} + k_f \frac{\partial \vec{B}}{\partial t} = 0 \qquad (10)$$

And finally we have the absence of magnetic monopoles

$$\vec{\nabla}.\vec{B} = 0 \tag{16}$$

Equations (10), (11), (15) and (16) are the Maxwell's equations in vacuum. These include all the constants, k_e, k_m, k_B, k_f and therefore are valid in all systems of units. As we said earlier, there is a relation between k_B and k_f which can now be obtained. In the absence of sources ($\rho = 0, \vec{J} = 0$), equation (15) becomes

$$\vec{\nabla} \times \vec{B} - \frac{k_B k_m}{k_e} \frac{\partial \vec{E}}{\partial t} = 0$$

On taking the curl of this equation and using the other Maxwell's equations leads to the wave equation

$$\nabla^2 \vec{B} - \frac{k_f k_B k_m}{k_e} \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

This is the equation of a wave propagating with a velocity $\sqrt{\frac{k_e}{k_B k_m k_f}}$. Knowing experimentally that electromagnetic waves travel with speed c in vacuum, we have

$$\sqrt{\frac{k_e}{k_B k_m k_f}} = c. \tag{17}$$

But we already know from equation (4) that $\sqrt{\frac{k_e}{k_m}} = c$. Hence

$$k_B k_f = 1 \tag{18}$$

The product $k_B k_f$ is dimensionless and numerically equals unity. Therefore, only two of the four constants are independent. Let us choose k_e and k_f as independent and write the other two in terms of these by using equations (4) and (18). Maxwell's equations then take the form

$$\vec{\nabla}.\vec{E} = 4\pi k_e \rho \tag{19}$$

$$\vec{\nabla} \times \vec{B} - \frac{k_B}{c^2} \frac{\partial \vec{E}}{\partial t} = 4\pi \frac{k_B k_e}{c^2} \vec{J} \tag{20}$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{k_B} \frac{\partial \vec{B}}{\partial t} = 0 \tag{21}$$

$$\vec{\nabla}.\vec{B} = 0 \tag{22}$$

The various systems differ in their choice of numerical values and dimensions of the remaining constants k_e and k_B. We list below the values of the four parameters for the Gaussian, the SI units that are currently widely in use, and some other systems which have been in use in the past and are still used sometimes.

System	k_e	$k_m = k_e/c^2$	k_B	$k_f = 1/k_B$
Electrostatic (esu)	1	$1/c^2$	1	1
Electromagnetic (emu)	c^2	1	1	1
Heaviside-Lorentz	$\frac{1}{4\pi}$	$\frac{1}{4\pi c^2}$	c	c^{-1}
Gaussian	1	$\frac{1}{c^2}$	c	c^{-1}
SI (MKSA)	$\frac{1}{4\pi\varepsilon_0} = 10^{-7}c^2$	$\frac{1}{4\pi\varepsilon_0 c^2} \equiv \frac{\mu_0}{4\pi} = 10^{-7}$	1	1

The system of units used for mechanical quantities along with the electromagnetic quantities is also different for different systems enumerated above. Whereas the SI system listed above uses the SI system (MKS) for mechanical quantities, all others use the cgs (centimeter, gram, second) system. Also, the SI system has four fundamental dimensions – length, mass, time and current, all others have only three – length, mass and time. All electrical and magnetic quantities are expressed in terms of the mechanical quantities only. For example, in Gaussian system, Coulomb's law takes the form

$$F_e = \frac{q_1 q_2}{r^2} \Rightarrow [q] = [M^{1/2} L^{3/2} T^{-1}]$$

Similarly for all other electromagnetic quantities.

35.3.1 Maxwell's equations in a medium

So far we have discussed the electromagnetic fields in free space only. We now wish to extend the discussion to fields in a medium. In a medium apart from the fields due to free charges and currents, fields also arise due polarization, $\vec{P}$ (electric dipole moment per unit volume) and magnetization, $\vec{M}$ (magnetic dipole moment per unit volume) of the medium. We account for these additional fields, which may be intrinsic to the medium or may be the result of external fields, by defining the *auxiliary fields*, $\vec{D}$ and $\vec{H}$, through the *constitutive relations*

$$\vec{D} = \varepsilon_0 \vec{E} + \lambda \vec{P}$$
$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \lambda' \vec{M} \qquad (23)$$

$\vec{D}$ is called the *displacement vector*, and $\vec{H}$ is called the *magnetic field intensity*. Four new constants have appeared in these equations. They will have different values and dimensions in different systems. Dimensions of $\vec{P}$ and $\vec{M}$ are known from their definitions.

$$[\vec{P}] = [\frac{\vec{p}}{L^3}] = [QL^{-2}]$$
$$[\vec{M}] = [\frac{\vec{m}}{L^3}] = [\frac{IL^2}{L^3}] = [IL^{-1}]$$

Nothing is gained by making $\vec{D}$ and $\vec{P}$ or $\vec{H}$ and $\vec{M}$ have different dimensions. So we choose λ and λ' to be dimensionless numbers with numerical value unity in Lorentz-Heaviside and SI system and 4π in the others mentioned above.

For linear, isotropic media, the constitutive relations are written as

$$\vec{D} = \varepsilon \vec{E}$$
$$\vec{H} = \frac{1}{\mu}\vec{B} \qquad (24)$$

Thus in equation (23) the constants ε_0, μ_0 are the vacuum values of ε, μ.

The continuity equation and the Ohms's law take the same form in all the systems enumerated above, viz.

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0; \quad \vec{J} = \sigma \vec{E}$$

35.4 Conversion Table

The two main systems which are still quite widely used in the literature are the Gaussian and the SI systems. Equations appear quite different in the two systems and it will be useful to be able to convert equations from one system to the other. The table below lists the various conversion factors required to obtain equations in one system from those written in the other.

	Quantity	Gaussian	SI
1	Velocity of light	c	$(\mu_0\varepsilon_0)^{-1/2}$
2	Electric field, potential, voltage	$\vec{E},\Phi,V$	$(4\pi\varepsilon_0)^{1/2}[\vec{E},\Phi,V]$
3	Displacement	$\vec{D}$	$(4\pi/\varepsilon_0)^{1/2}\vec{D}$
4	Charge, charge density, current, current density, polarization	$\rho,q,I,\vec{J},\vec{P}$	$(4\pi\varepsilon_0)^{-1/2}[\rho,q,I,\vec{J},\vec{P}]$
5	Magnetic induction	$\vec{B}$	$(\frac{4\pi}{\mu_0})^{1/2}\vec{B}$
6	Magnetic field intensity	$\vec{H}$	$(4\pi\mu_0)^{1/2}\vec{H}$
7	Magnetization	$\vec{M}$	$(\frac{\mu_0}{4\pi})^{1/2}\vec{M}$
8	Conductivity	σ	$\frac{1}{4\pi\varepsilon_0}\sigma$
9	Dielectric constant	ε	$\frac{1}{\varepsilon_0}\varepsilon$
10	Permeability	μ	$\frac{1}{\mu_0}\mu$
11	Resistance, impedence, inductance	R, Z, L	$(4\pi\varepsilon_0)[R,Z,L]$
12	Capacitance	C	$\frac{1}{4\pi\varepsilon_0}C$

The factor c appears explicitly in equations written in Gaussian system. The first entry means that wherever the factor of c appears, replace it by $(\mu_0\varepsilon_0)^{-1/2}$ to obtain the corresponding equation in SI system. To convert equations written in the SI system into Gaussian, first make proper replacements for all electromagnetic quantities. If any factor of $(\mu_0\varepsilon_0)$ is left, replace it with $1/c^2$.

Example-1

The Lorentz force equation in SI system is

$$\vec{F} = q[\vec{E} + \vec{v} \times \vec{B}]$$

To obtain the equation in Gaussian system, replace q by $(4\pi\varepsilon_0)^{1/2} q$, $\vec{E}$ by $(4\pi\varepsilon_0)^{-1/2}\vec{E}$ and $\vec{B}$ by $(\frac{\mu_0}{4\pi})^{1/2}\vec{B}$. Hence, in Gaussian units

$$\vec{F} = q[\vec{E} + \sqrt{\varepsilon_0\mu_0}\vec{v}\times\vec{B}]$$

Or

$$\vec{F} = q[\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}]$$

Example-2

In Gaussian units Ampere's law with modification due to Maxwell reads

$$\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

To obtain the equation in the SI system, make the following replacements:

$$\vec{H}\rightarrow(4\pi\mu_0)^{1/2}\vec{H}; \qquad c\rightarrow(\mu_0\varepsilon_0)^{-1/2};$$

$$\vec{D}\rightarrow(4\pi/\varepsilon_0)^{1/2}\vec{D}; \qquad \vec{J}\rightarrow(4\pi\varepsilon_0)^{-1/2}\vec{J}$$

Hence in SI units, the equation becomes

$$(4\pi\mu_0)^{1/2}\vec{\nabla}\times\vec{H} - \frac{1}{(\mu_0\varepsilon_0)^{-1/2}}(\frac{4\pi}{\varepsilon_0})^{1/2}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{(\mu_0\varepsilon_0)^{-1/2}}(4\pi\varepsilon_0)^{-1/2}\vec{J}$$

Or

$$\vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t} = \vec{J}$$

Example-3

Faraday's law in SI unit reads

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

To write the same law in Gaussian system, replace $\vec{E} \rightarrow (4\pi\varepsilon_0)^{-1/2}\vec{E}$ and $\vec{B} \rightarrow (\frac{\mu_0}{4\pi})^{1/2}\vec{B}$, so in Gaussian system the equation becomes

$$(4\pi\varepsilon_0)^{-1/2}\vec{\nabla} \times \vec{E} + (\frac{\mu_0}{4\pi})^{1/2}\frac{\partial \vec{B}}{\partial t} = 0$$

Or

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

Summary

1. *In electromagnetism, a vast number of different systems of units and dimensions have been used in the past. Reasons for so many systems of units and dimensions in electromagnetism are explained.*
2. *Various equations of electromagnetism are written in a system-independent form by introducing four constants of proportionality that appear in these equations.*
3. *It is explained that of the four, only two are really independent and how giving different values and dimensions to these constants lead to different systems.*
4. *The values of these constants are tabulated for five of the many systems that had been in use in the past. By and large only two of those systems are in use now, Gaussian and SI. The table of course includes these two.*
5. *The equations appear quite different in these two systems leading to confusion to students and scholars as well. A table is provided for conversion of equations from one system to the other and explained with examples.*